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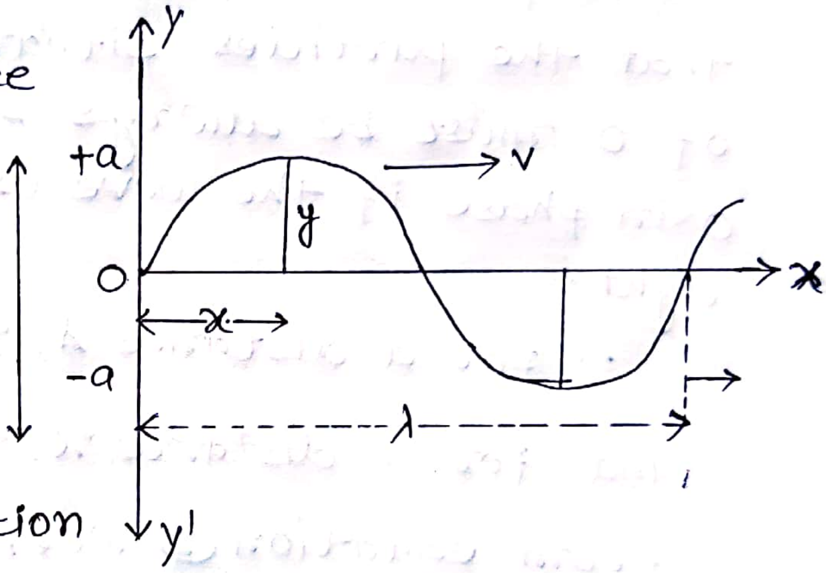
B.Sc (Part-1) Hons
Paper-1

Equation of progressive and Stationary waves

Progressive wave:—

A plane progressive harmonic wave is the simplest wave in which the particles of the medium perform simple harmonic motion.

Let the source of a simple harmonic motion disturbance be situated at the origin O, and the wave travels in the positive direction x-axis in a continuous homogeneous medium.



As the wave proceeds onward, each successive particle of the medium is set in simple harmonic oscillation. In the figure shows the displacement (y) of particles against their positions (x) at a particular instant t. If we measure time from the instant when the particle at O is passing through its mean position, then the displacement of this particle at any instant t along y-axis is given by

$$y = a\sin\omega t = a\sin\left(\frac{2\pi t}{T}\right) \quad \text{——— (1)}$$

where 'a' is the amplitude of the particle and T the periodic time taken by the wave to cover a distance λ.

The displacement at the same instant of a particle p at a distance x from o towards the right can be written as

$$y = a\sin(\omega t - \phi) \quad \text{——— (2)}$$

where ϕ is the phase lag. Since, it is evident that the particles situated towards the right of o must be always behind the particle at o in phase if the wave is travelling towards the right.

$\because$ For a distance λ, the phase lag $= 2\pi$

and for a distance x, the phase lag $\phi = \frac{2\pi}{\lambda}x$

$\therefore$ From equation (2) we have,

$$y = a\sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right)$$

$$= a\sin\left(\frac{2\pi vt}{\lambda} - \frac{2\pi x}{\lambda}\right) \quad \left(\because v = \frac{\lambda}{T}\right)$$

$$\therefore \boxed{y = a\sin\frac{2\pi}{\lambda}(vt - x)} \quad \text{——— (3)}$$

This is the equation of progressive waves.

Stationary waves:— When two identical waves either transverse or longitudinal, travel through a medium along the same line in opposite directions, they superimpose to produce a new type of waves which appear stationary in space.

These waves are called 'Stationary waves' or 'Standing waves' For example, when a waves is sent along a string or along the air column of a pipe, it is reflected at the other end and superimpose upon the incident wave to produce stationary waves.

The equation of a plane progressive wave of amplitude a travelling with a velocity v along positive x-direction is given by

$$y_1 = a \sin \frac{2\pi}{\lambda}(vt - x).$$

Where y_1 is the displacement at any point x at an instant t. The wavelength is λ.

Let us assume that this wave is reflected from a rigid boundary wall situated at $x=0$. The displacment at any point x due to the reflected wave moving along x-direction at an instant t will be given by

$$y_2 = a' \sin \frac{2\pi}{\lambda}(vt + x)$$

where a' is the amplitude of the reflected wave. By the principle of superposition, the resultant displacement at the point x at the same instant t will be given by

$$y = y_1 + y_2$$

$$= a \sin \frac{2\pi}{\lambda}(vt - x) + a' \sin \frac{2\pi}{\lambda}(vt + x) \quad \text{—} (4)$$

The displacement y at the rigid wall at $x=0$ will always be zero.

$\therefore\ y = 0 = a\sin\left(\frac{2\pi vt}{\lambda}\right) + a'\sin\left(\frac{2\pi vt}{\lambda}\right)$

or, $(a+a')\sin\left(\frac{2\pi vt}{\lambda}\right) = 0$

$\therefore\ a + a' = 0$ or, $a = -a'$

From equation (4), we have

$$y = a\left[\sin\frac{2\pi}{\lambda}(vt-x) - \sin\frac{2\pi}{\lambda}(vt+x)\right]$$

or, $y = a\left[2\cos\frac{2\pi vt}{\lambda}\left\{\sin\left(-\frac{2\pi x}{\lambda}\right)\right\}\right]$

or, $y = -2a\cos\left(\frac{2\pi vt}{\lambda}\right)\cdot\sin\left(\frac{2\pi x}{\lambda}\right)$

or, $\boxed{y = -2a\sin\left(\frac{2\pi x}{\lambda}\right)\cdot\cos\left(\frac{2\pi vt}{\lambda}\right)}$ —— (5)

This represents a simple harmonic vibration of amplitude $\left[-2a\sin\left(\frac{2\pi x}{\lambda}\right)\right]$ and of the same period as the original waves.

This is the equation of the resultant stationary waves.

Differential equation of wave motion:—

The equation of plane progressive wave is

$$y = a\sin\frac{2\pi}{\lambda}(vt-x) \quad \text{—— (6)}$$

$\therefore$ particle velocity $= \frac{dy}{dt} = \left(\frac{2\pi v}{\lambda}\right)a\cos\frac{2\pi}{\lambda}(vt-x)$ —— (7)

Differentiating equation (6) w.r.t. x, we have

$$\frac{dy}{dx} = -\left(\frac{2\pi}{\lambda}\right)a\cos\frac{2\pi}{\lambda}(vt-x) \quad \text{—— (8)}$$

$\frac{dy}{dx}$ measure the slope of the displacement curve at the point x.

Dividing equation (7) by equation (8), we have

$$\frac{dy}{dt} = -v\frac{dy}{dx} \quad \text{—— (9)}$$

The acceleration of the particle is

$$f = \frac{d^2y}{dt^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 a \sin\frac{2\pi}{\lambda}(vt-x) \quad \text{—— (10)}$$

Again differentiating equation (8) with respect to x, we have

$$\frac{d^2y}{dx^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \sin\frac{2\pi}{\lambda}(vt-x) \quad \text{—— (11)}$$

Dividing equation (10) by equation (11) we get

$$\boxed{\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}}$$

This equation called the differential equation of wave motion.

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